QUANTUM SEMIGROUPS GENERATED BY LOCALLY COMPACT SEMIGROUPS

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Abstract. Let S be a subsemigroup of a locally compact group G, such that $S^{-1}S = G$. We consider the C^*-algebra $C^*_\delta(S)$ generated by the operators of translation by all elements of S in $L^2(S)$. We show that this algebra admits a comultiplication which turns it into a compact quantum semigroup. For S with nonempty interior, this comultiplication can be extended to the von Neumann algebra $VN(S)$ generated by $C^*_\delta(S)$.

1. Introduction

The notion of a quantum semigroup, as a C^*- or von Neumann algebra with a comultiplication, appeared well before the term and before the notion of a locally compact quantum group. But it is especially these last years that substantial examples of quantum semigroups are considered; we would like to mention families of maps on finite quantum spaces [17], quantum semigroups of quantum partial permutations [3], quantum weakly almost periodic functionals [9], quantum Bohr compactifications [16, 18].

In this article we construct a rather "classical" family of compact quantum semigroups, which are associated to sub-semigroups of locally compact groups. The interest of our objects is in fact that they provide natural examples of C^*-bialgebras which are co-commutative and are not however duals of functions algebras. Recall that the classical examples of quantum groups belong to one of the two following types: they are either function algebras, such as the algebra $C_0(G)$ of continuous functions vanishing at infinity on a locally compact group G, or their duals, such as the reduced group C^*-algebras $C^*_r(G)$. In the semigroup situation one can go beyond this dichotomy.

If S is a discrete semigroup, then the algebra $C^*_\delta(S)$ which we consider coincides with the reduced semigroup C^*-algebra $C^*_r(S)$ which has been known since long ago [6, 7, 2, 20, 15]. If $S = G$ is a locally compact group, then $C^*_\delta(S) = C^*_\delta(G)$ is the C^*-algebra generated by all left translation operators

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in $B(L^2(G))$ [12, 4]. If G is moreover abelian, then $C^*_\delta(G)$ equals to the algebra $C(\widehat{G}_d)$ of continuous functions on the dual of the discrete group G_d [12].

The new case considered in this paper concerns non-discrete nontrivial subsemigroups of locally compact groups, and our objective is to show that they admit a natural coalgebra structure. Let G be a locally compact group, and let S be its sub-semigroup such that $S^{-1}S = G$. Set $H_S = \{f \in L^2(G) : \mathrm{supp} f \subset S\}$; let E_S be the orthogonal projection of $L^2(G)$ onto H_S and let J_S be the right inverse of E_S, so that $E_S J_S = \mathrm{Id}_{H_S}$. After the study of semigroup ideals in Section 2, in Section 3 we define $C^*_\delta(S)$ as the C^*-algebra generated in $B(H_S)$ by the operators $T_a = E_S L_a J_S$ over all $a \in S$, where L_a is the operator of the left translation by a on $L^2(G)$. In Section 4 we show that $C^*_\delta(S)$ admits a comultuplication Δ such that $\Delta(T_a) = T_a \otimes T_a$. The same facts are derived in parallel for the universal semigroup C^*-algebra $C^*(S)$ defined axiomatically via defining relations.

In the discrete case, the construction was carried out by X. Li [13] (see also [8]). The discrete abelian case was studied in detail in [1].

In Section 5, we show that for S with nonempty interior, the comultiplication can be extended to the von Neumann algebra $VN(S)$ generated by $C^*_\delta(S)$ in $B(H_S)$. In the case $S = G$, this is the classical group von Neumann algebra. We can thus call $VN(S)$ the semigroup von Neumann algebra.

2. Semigroup ideals

Let G be a locally compact group, S a closed subsemigroup of G containing the identity e of G and such that $G = S^{-1}S$. Denote by μ the left Haar measure on G.

For any subset $X \subset S$ and any $p \in S$, define the *translations in* S:

$$pX = \{pq \colon q \in X\},\ p^{-1}X = \{q \in S \colon pq \in X\}. \tag{2.1}$$

Obviously, pS is a right ideal in S, $eS = e^{-1}S = S$ and $p^{-1}S = S$ for any $p \in S$. For the usual translations in G, we use the notation $g \cdot_G X = \{gh : h \in G\}$, so that $p^{-1}X = S \cap p^{-1} \cdot_G X$. It is also easy to see that $p(qX) = (pq)X$ and $p^{-1}(q^{-1}X) = (qp)^{-1}X$ for any $X \subset S$ and all $p, q \in S$. We will omit parentheses in the products of this type. Moreover, $p^{-1}pX = X$, but in general, the products pq^{-1} or $p^{-1}q$ should be viewed purely formally, and $pp^{-1}X$ might differ from X (see, for example, Lemma 2.2).

More precisely, denote by $\mathcal{F} = \mathcal{F}(S)$ the free monoid generated by S and $S^{-1} \setminus S$. Any element in $\mathcal{F}$ is a finite word with alternating symbols in S and S^{-1}. The operation of taking inverse in G induces the operation $w \mapsto w^{-1}$ on the monoid $\mathcal{F}$, by $\left(p_1^{\pm 1} \cdots p_n^{\pm 1}\right)^{-1} = p_n^{\mp 1} \cdots p_1^{\mp 1}$.

For every $w = p_1^{\pm 1} \cdots p_n^{\pm 1} \in \mathcal{F}$ and $X \subset S$, define by induction $wX = p_1^{\pm 1}(\ldots(p_n^{\pm 1}X)\ldots)$. If $X = S$, then wS is a right ideal in S. Define the

family of all *constructible right ideals* in S [13]:

$$\mathcal{J} = \{\bigcap_{i=1}^{n} w_i S \colon w_i \in \mathcal{F}\} \cup \{\emptyset\}.$$

Suppose that $w \in F$ has the form $w = p_1^{-1}q_1p_2^{-1}q_2 \dots p_n^{-1}q_n$ with $p_j, q_j \in S$, maybe with $p_1 = e$ or $q_n = e$. Then it follows from the definition that wS is the set of elements x satisfying

$$\begin{gathered} x = p_1^{-1}q_1 \dots p_n^{-1}q_n r_{n+1}, \\ \text{where } r_{n+1} \in S \text{ and} \\ r_k = p_k^{-1}q_k \dots p_n^{-1}q_n r_{n+1} \in S \text{ for all } k = 1, \dots, n. \end{gathered} \tag{2.2}$$

Define a homomorphism $\mathcal{F} \to G$: $w \mapsto (w)_G$, by $(p^{\pm 1})_G = p^{\pm 1}$ for $p \in S$. We fix also an injection $\imath : G \hookrightarrow \mathcal{F}$ which might not be a homomorphism: for any element $g \in G$ we fix one of its representations $g = p^{-1}q$ and set $\imath(g) = p^{-1}q \in \mathcal{F}$. The notation gX, where $g \in G$ and $X \subset S$, is understood in the sense $gX = \imath(g)X$.

Lemma 2.1. *For any $w_1, w_2 \in \mathcal{F}$, we have $w_1w_2S \subset w_1S$.*

Proof. Follows immediately from the facts that $pS \subset S, p^{-1}S = S$ for any $p \in S$. □

Lemma 2.2. *For any $w \in \mathcal{F}$, $wS = ww^{-1}S$.*

Proof. We can assume that w has the form $w = p_1^{-1}q_1p_2^{-1}q_2 \dots p_n^{-1}q_n$ with $p_j, q_j \in S$, maybe with $p_1 = e$ or $q_n = e$. Then every $x \in wS$ has the form (2.2) with $r_{n+1} \in S$ and $r_k = p_k^{-1}q_k \dots p_n^{-1}q_n r_{n+1} \in S$ for all $k = 1, \dots, n$.

Now write $x = p_1^{-1}q_1 \dots p_n^{-1}q_n q_n^{-1}p_n \dots q_1^{-1}p_1 x$. In this product, $x \in S$ and $r_{k+1} = q_k^{-1}p_k \dots q_1^{-1}p_1 x \in S$ for $k = 1, \dots, n$, as well as $r_k = p_k^{-1}q_k \dots p_n^{-1}q_n q_n^{-1}p_n \dots q_1^{-1}p_1 x \in S$ for $k = 1, \dots, n$. It follows that $x \in ww^{-1}S$, so $wS \subset ww^{-1}S$. The inverse inclusion follows from Lemma 2.1. □

Lemma 2.3. *Let a word $w \in \mathcal{F}$ have the form $w = w_1w_2$, where $w_1, w_2 \in \mathcal{F}$. Then $wS = w_1S \cap (w_1)_G w_2 S$.*

Proof. Suppose $w_1 = p_1^{-1}q_1p_2^{-1}q_2 \dots p_i^{-1}q_i$, $w_2 = p_{i+1}^{-1}q_{i+1}p_{i+2}^{-1}q_{i+2} \dots p_n^{-1}q_n$ with $p_j, q_j \in S$. Then every $x \in wS$ satisfies (2.2). This implies directly that $x \in w_1S$. If we denote $(w_1)_G = p^{-1}q$, $p, q \in S$, then in the notations (2.2) we have also $x = p^{-1}qr_{i+1}$ what implies that $x \in p^{-1}qw_2S = wS$.

Conversely, if $wS = w_1S \cap (w_1)_G w_2S$, then

$$\begin{gathered} x = p_1^{-1}q_1p_2^{-1}q_2 \dots p_i^{-1}q_i r'_{i+1}, \\ r'_{i+1} \in S,\ r'_k = p_k^{-1}q_k \dots p_i^{-1}q_i r'_i \in S \quad \text{for } k = 1, \dots, i, \end{gathered}$$

and

$$x = p^{-1}qp_{i+1}^{-1}q_{i+1}p_{i+2}^{-1}q_{i+2} \dots p_n^{-1}q_n r_{n+1},$$

$$x \in S,\ r_{n+1} \in S,\ r_k = p_k^{-1}q_k \dots p_n^{-1}q_n r_{n+1} \in S \quad \text{for } k = i+1, \dots, n.$$

By cancellation, it follows that $r'_{i+1} = p_{i+1}^{-1}q_{i+1}p_{i+2}^{-1}q_{i+2} \dots p_n^{-1}q_n r_{n+1}$, thus in fact the condition (2.2) holds for x. □

Corollary 2.4. *For any $v, w \in \mathcal{F}$, we have $vS \cap wS = ww^{-1}vS$.*

Proof. Since $(ww^{-1})_G = e$, by Lemmas 2.2 and 2.3 we have

$$wS \cap vS = ww^{-1}S \cap vS = ww^{-1}S \cap (ww^{-1})_G vS = ww^{-1}vS.$$

□

It follows that

$$\mathcal{J} = \{wS|\ w \in F\} \cup \{\emptyset\}.$$

3. The semigroup C*-algebras

In what follows we assume the following property for S. If $X = \cup_{j=1}^n X_j$ up to a null set for $X, X_1, \dots, X_n \in \mathcal{J}$, then $X = X_j$, also up to a null set, for some $1 \leq j \leq n$. We call this property *topological independence of constructible right ideals* in S, the algebraic one being introduced in [13]. It is exactly this property which will guarantee that our comultiplication is well defined. Note that the topological independence implies the algebraic one.

Recall the definition of the full semigroup C*-algebra [13]. Consider a family of isometries $\{v_p|p \in S\}$ and a family of projections $\{e_X|X \in \mathcal{J}\}$ satisfying the following relations for any $p, q \in S$, and $X, Y \in \mathcal{J}$:

$$v_{pq} = v_p v_q,\ v_p e_X v_p^* = e_{pX}, \tag{3.1}$$

$$e_S = 1,\ e_\emptyset = 0,\ e_{X\cap Y} = e_X e_Y. \tag{3.2}$$

The universal C*-algebra $C^*(S)$ of the semigroup S is by definition generated by $\{v_p|p \in S\} \bigcup \{e_X|X \in \mathcal{J}\}$ with the relations above.

The C^*-algebra $C^*(S)$ contains a commutative C^*-algebra $D(S)$ generated by the family of projections $\{e_X|X \in \mathcal{J}\}$.

Consider the Hilbert space $L^2(G)$ with respect to μ. For any measurable subset $X \subset G$ set $H_X = \{f \in L^2(G) : \text{esssupp} f \subset X\}$; this subspace is isomorphic to $L^2(X, \mu)$. Let $I_X \in L^2(G)$ be the characteristic function of X and E_X the orthogonal projection of $L^2(G)$ onto H_X, which is just the multiplication by I_X. Let $L\colon G \to B(L^2(G))$ be the left regular representation of G, i.e. for any $a, b \in G,\ f \in L^2(G)$

$$(L_a f)(b) = f(a^{-1}b). \tag{3.3}$$

We define the left regular representation $T\colon S \to B(H_S)$ of the semigroup S analogously to L. For any $a, b \in S,\ f \in H_S$ we set

$$(T_a f)(b) = f(a^{-1}b), \tag{3.4}$$

so that $T_p = E_S L_p E_S$; then

$$(T_a^* f)(b) = I_S(b) f(ab). \tag{3.5}$$

One can verify that T_a is an isometry, $T_a^* T_a = I$, and that for any $f \in H_S$ for $a, b \in S$ we have

$$(T_a T_a^* f)(b) = I_S(a^{-1}b) f(b).$$

Clearly, $a^{-1}b \in S$ if and only if $b \in aS$, where aS is a constructible right ideal defined in the previous section. Hence the projection $T_a T_a^*$ is an operator of multiplication by I_{aS}. The map $T\colon S \to B(H_S)$ is obviously a representation of S. Let $C^*_\delta(S)$ be the C^*-subalgebra in $B(H_S)$ generated by the operators $\{T_a : a \in S\}$.

If $S = G$, then $C^*_\delta(S) = C^*_\delta(G)$ is the C^*-algebra generated by all left translation operators in $B(L^2(G))$ [12, 4]. If S is discrete, then $C^*_\delta(S) = C^*_r(S)$ is the reduced semigroup C^*-algebra [15].

A finite product of the generators T_a, T_b^* for any $a, b \in S$ is called *a monomial*. We will use also the notation $T_{a^{-1}} = T_a^*$, what does not create confusion in the case $a^{-1} \in S$. Generally, for every $w = p_1^{\pm 1} \dots p_n^{\pm 1} \in \mathcal{F}$ we denote $T_w = T_{p_1^{\pm 1}} \dots T_{p_n^{\pm 1}}$, and clearly every monomial has this form.

Lemma 3.1. *For any monomial T_w, function $f \in H_S$ and $x \in G$ we have*

$$(T_w f)(x) = I_{wS}(x) \cdot f((w^{-1})_G x) \tag{3.6}$$

Proof. Let k be the length of the word w. For $k = 1$, either $w = a$ or $w = a^{-1}$ with some $a \in S$. If $w = a$ then for $f \in H_S$ we have $f(a^{-1}x) = f(a^{-1}x) I_S(a^{-1}x) = I_{aS}(x) f(a^{-1}x)$, thus the expressions (3.4) and (3.6) are equal. If $w = a^{-1}$, then, due to the fact that $a^{-1}S = S$, the formula (3.5) implies (3.6).

Suppose (3.6) is proved for $k \le n$ and $w = vw'$ is a word in $\mathcal{F}$ with the length $k+1$, where the length of v and w' is 1 and k respectively. First assume that $v = a \in S$ and denote $g = T_{w'} f$. Then for any $x \in G$ we have

$$\begin{aligned}(T_w f)(x) &= (T_a T_{w'} f)(x) = (T_a g)(x) = \\ &= g(a^{-1}x) = (T_{w'} f)(a^{-1}x) = \\ &= I_{w'S}(a^{-1}x) f((w'^{-1})_G a^{-1} x) = \\ &= I_{aw'S}(x) f(((aw')^{-1})_G x).\end{aligned}$$

Now assume that $v = a^{-1} \in S^{-1}$. Then for any $x \in G$ we have

$$(T_w f)(x) = (T_a^* T_{w'} f)(x) = (T_a^* g)(x) =$$
$$= I_S(x) g(ax) = I_S(x)(T_{w'} f)(ax) = I_S(x) I_{w'S}(ax) f((w'^{-1})_G ax) =$$

Note that $x \in S$ and $ax \in w'S$ if and only if $x \in a^{-1}w'S$.

$$= I_{a^{-1}w'S}(x) f(((a^{-1}w')^{-1})_G x) = I_{wS}(x) \cdot f((w^{-1})_G x).$$

And the formula (3.6) follows. $\square$

Lemma 3.2. *The C^*-algebra $C^*_\delta(S)$ is isomorphic to the C^*-subalgebra in $B(L^2(G))$ generated as a **linear space** by*

$$E_{wS}L_{(w)_G}E_S, \quad w \in \mathcal{F}, \tag{3.7}$$

and equivalently by

$$E_{wS}L_{(w)_G}, \quad w \in \mathcal{F}. \tag{3.8}$$

Proof. It follows directly from (3.6) that $T_wE_S = E_{wS}L_{(w)_G}E_S$ for every $w \in \mathcal{F}$. At the same time, $E_STE_S = TE_S$ for every $T \in C^*_\delta(S)$. Thus, the mapping $T \mapsto TE_S = E_STE_S$ is a *-homomorphism from $C^*_\delta(S)$ to $B(L^2(G))$, and its image is generated exactly by the operators (3.8). Moreover, this mapping is clearly isometric and thus it is an isomorphism.

To arrive at the second description, one calculates that $L_gE_S = E_{g\cdot_G S}L_g$ for every $g \in G$. Thus,

$$E_{wS}L_{(w)_G}E_S = E_{wS\cap((w)_G\cdot_G S)}L_{(w)_G}.$$

By definition, $wS \subset (w)_G \cdot_G S$, and (3.8) follows. □

For a monomial T_w define its *index* by $(w)_G \in G$. We have $\mathrm{ind}T^*_w = (w)^{-1}_G$ and $\mathrm{ind}(T_vT_w) = (v)_G(w)_G$. Recall that $E_X \in B(L^2(G))$ is the operator of multiplication by I_X.

Corollary 3.3. *A monomial T_w in $C^*_\delta(S)$ is an orthogonal projection if and only if $\mathrm{ind}T_w = e$. In this case $T_w = E_{wS}$.*

Proof. Let T_w be an orthogonal projection. Then $(ww)_G = (w)^2_G = (w)_G$ and $w^{-1}_G = w_G$. Hence, $(w)_G = \mathrm{ind}T_w = e$.

Suppose that $\mathrm{ind}T_w = e$. Then due to Lemma 3.1, $T_w = E_{wS}$ which is an orthogonal projection. □

Lemma 3.4. *Every projection E_X for $X \in \mathcal{J}$ is contained in $C^*_\delta(S)$ and equals $T_{ww^{-1}}$ for some $w \in \mathcal{F}$.*

Proof. By Corollary 2.4, $X = wS$ for some $w \in F$. Due to Corollary 3.3, if $(w)_G = e$ then $E_{wS} \in C^*_\delta(S)$.

Suppose w is an arbitrary element in $\mathcal{F}$. By Lemma 2.2, $wS = ww^{-1}S$ and $E_{wS} = E_{ww^{-1}S}$. Since $(ww^{-1})_G = e$, by Corollary 3.3 we have that $E_{wS} = T_{ww^{-1}} \in C^*_\delta(S)$. □

Lemma 3.5. *There exists a surjective *-homomorphism $\lambda\colon C^*(S) \to C^*_\delta(S)$ such that $\lambda(v_p) = T_p$, $\lambda(e_X) = E_X$. It will be called the left regular representation of $C^*(S)$.*

Proof. One can easily verify that the operators T_p and E_X satisfy the equations (3.1) and (3.2) for all $p \in S,\ X \in \mathcal{J}$. The universality of $C^*(S)$ implies the existence of the homomorphism λ. □

Denote by $D_\delta(S)$ the C^*-subalgebra in $C^*_\delta(S)$ generated by monomials with index equal to e. By Corollary 3.3 and Lemma 3.4 $D_\delta(S)$ is generated by projections $\{E_X|\ X \in \mathcal{J}\}$, and is obviously commutative.

Lemma 3.6. *The algebras $D(S)$ and $D_\delta(S)$ are isomorphic.*

Proof. The left regular representation restricted to $D(S)$ is surjective. Applying Lemma 2.20 in [13] and using the independence of constructible right ideals in S we obtain injectivity of $\lambda|_{D(S)}$. □

There exists a natural action of the semigroup S on the C^*-algebra $D_\delta(S)$.

$$\tau_p(A) = T_p A T_p^*,\ p \in S,\ A \in D_\delta(S). \tag{3.9}$$

Using the formula (3.6), we obtain for $A = E_X$, $X \in \mathcal{J}$:

$$\tau_p(E_X) = E_{pX}. \tag{3.10}$$

4. The universal and reduced compact quantum semigroups

Consider the C^*-subalgebra $\mathcal{A}$ in $C^*(S) \otimes_{max} C^*(S)$ generated by the elements

$$\{v_p \otimes v_p,\ e_X \otimes e_X : p \in S,\ X \in \mathcal{J}\}.$$

Clearly, these elements satisfy relations (3.1), (3.2). The universal property of $C^*(S)$ implies the existence of a unital $*$-homomorphism $\Delta_u : C^*(S) \to \mathcal{A}$, such that

$$\Delta_u(v_p) = v_p \otimes v_p,\ \Delta_u(e_X) = e_X \otimes e_X.$$

The map Δ_u admits a restriction $\Delta_u|_{D(S)} : D(S) \to D(S) \otimes_{max} D(S)$ which is also a unital *-homomorphism.

The pair $\mathbb{Q}(S) = (C^*(S), \Delta_u)$ is a compact quantum semigroup [1]. We call the algebra $C^*(S)$ with this structure the *universal algebra of functions on the compact quantum semigroup* $\mathbb{Q}(S)$ associated with the semigroup S.

We recall that a semigroup is called right reversible if every pair of non-empty left ideals has a non-empty intersection. The following theorem by Ore can be found in [5].

Theorem 4.1. *A cancellative semigroup S can be embedded into a group G such that $G = S^{-1}S$ if and only if it is right reversible.*

Define a partial order on S: $p \leq q$ if $qp^{-1} \in S$, or equivalently if $q \in Sp$. Due to the assumption $G = S^{-1}S$, we obtain by Theorem 4.1 that for any $p, q \in S$ the left ideals $\{xp : x \in S\}$, $\{yq : y \in S\}$ have a non-empty intersection. Hence S is upwards directed with respect to this partial order.

Consider the directed system of C^*-algebras $\mathcal{A}_p$ indexed by $p \in S$, where every $\mathcal{A}_p = D_\delta(S)$. For $p, q \in S$ such that $p \leq q$ we have $qp^{-1} \in S$ and the action (3.9) generates a $*$-homomorphism $\tau_{qp^{-1}} : \mathcal{A}_p \to \mathcal{A}_q$:

$$\tau_{qp^{-1}}(A) = T_{qp^{-1}} A T^*_{qp^{-1}}.$$

Clearly, $\tau_{qp^{-1}} = \tau_{qr^{-1}}\tau_{rp^{-1}}$ for $p \le r \le q$. Let $D_\delta^{(\infty)}(S)$ denote the C^*-inductive limit of the directed system $\{\mathcal{A}_p, \tau_{qp^{-1}}\}$.

Recall the notation $q^{-1} \cdot_G X = \{q^{-1}x \colon x \in X\} \subset G$ for $q \in S$ and $X \in \mathcal{J}$.

Lemma 4.2. *The C^*-algebra $D_\delta^{(\infty)}(S)$ is isomorphic to*

$$D = C^*(\{E_{q^{-1}\cdot_G X} \colon q \in S,\ X \in \mathcal{J}\}) \subset B(L^2(G)).$$

Proof. By definition, $D_\delta(S) \subset B(H_S)$. Recall that we denote $J_S : H_S \to L^2(G)$ the canonical imbedding; denote by $\pi \colon D_\delta(S) \to B(L^2(G))$ the lifting $\pi(A) = J_S A E_S$.

For any $p \in S$, the map

$$\phi_p(A) = L_p^* \pi(A) L_p, \quad A \in D_\delta(S),$$

is a $*$-homomorphism $\phi_p \colon D_\delta(S) \to D$, such that $\phi_p(E_X) = E_{p^{-1}\cdot_G X}$ for all $X \in \mathcal{J}$.

Then for $p \le q$ and $X \in \mathcal{J}$ we have:

$$\phi_q \tau_{qp^{-1}}(E_X) = L_q^* E_{qp^{-1}X} L_q = E_{q^{-1}\cdot(qp^{-1}X)} = E_{p^{-1}\cdot X} = \phi_p(E_X).$$

So the maps ϕ_p agree with $\tau_{qp^{-1}}$. The homomorphisms ϕ_p are injective since π is obviously injective and L_p is a unitary operator. It follows that the limit map $\Phi = \lim \phi_p \colon D_\delta^{(\infty)}(S) \to D$ is injective.

To prove surjectivity of Φ it suffices to show that for any $q_1, \ldots, q_n \in S$, $X_1, \ldots, X_n \in \mathcal{J}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ we have

$$\sum_i \lambda_i E_{q_i^{-1}\cdot_G X_i} \in \Phi(D_\delta^{(\infty)}(S)).$$

Since the system $\{\mathcal{A}_p, \tau_{qp^{-1}}\}$ is upwards directed, there exists $s \in S$ such that $q_i \le s$, $i = 1, 2, \ldots n$, and it implies that $sq_i^{-1}X_i \in \mathcal{J}$ and $q_i^{-1} \cdot_G X_i = s^{-1} \cdot_G (sq_i^{-1}X_i) \in \phi_s(D_\delta(S))$. Hence

$$\sum_i \lambda_i E_{q_i^{-1}\cdot_G X_i} \in \phi_s(D_\delta(S))$$

and we obtain

$$D = \overline{\bigcup_{p\in S} \phi_p(D_\delta(S))}$$

Therefore Φ is surjective and we get the isomorphism $D_\delta^{(\infty)}(S) \cong D$. $\square$

On $B(L^2(G))$, we have the adjoint action of G generated by the left regular representation: $A \mapsto L_g A L_g^*$. Let us show that D is invariant under this action. For $g \in G$

$$L_g E_{q^{-1}\cdot_G X} L_g^* = E_{(gq^{-1})\cdot_G X}.$$

Since $G = S^{-1}S$, we can write $gq^{-1} = t^{-1}s$ with some $s, t \in S$. Then $s \cdot_G X = s \cdot X \in \mathcal{J}$, and $E_{(gq^{-1})\cdot_G X} = E_{(t^{-1}s)\cdot_G X} \in D$.

The isomorphism Φ defined above allows then to define an action of G on $D_\delta^{(\infty)}(S)$: for $u \in D_\delta^{(\infty)}(S)$, $g \in G$ we set

$$\tau(g)u = \Phi^{-1}(L_g \Phi(u) L_g^*). \tag{4.1}$$

$(D_\delta^{(\infty)}(S), G, \tau)$ is a C^*-dynamical system, and by definition the pair (Φ, L) is a covariant representation of this system.

Lemma 4.3. *The reduced crossed product $D_\delta^{(\infty)}(S) \rtimes_{r,\tau} G$ of the commutative C^*-algebra $D_\delta^{(\infty)}(S)$ and the group G by the action τ, generated by the covariant representation (Φ, L), is isomorphic to $C^*(\{E_X, L_g \colon X \in S^{-1}\cdot\mathcal{J}, g \in G\})$.*

Proof. The integrated form $\Phi \times L$ of the covariant representation (Φ, L) is a faithful representation of $C_c(G, D_\delta^{(\infty)}(S))$ on $L^2(G)$ and can be extended to a covariant representation $(\Phi\times L)\otimes 1$ on $L^2(G)\otimes L^2(G) = L^2(G\times G)$. Consider a unitary operator $W \in B(L^2(G)\otimes L^2(G))$, given by $W\xi(x, y) = \xi(x, xy)$ for any $\xi \in L^2(G) \otimes L^2(G)$. We claim that W provides unitary equivalence between the representations $(\Phi \times L)\otimes 1$ and $(\Phi \times L)\otimes L$. To show this it is sufficient to check the equation for the components of the representations on the generators of the algebra $D_\delta^{(\infty)}(S)$ and of the group G. Denote $\Phi^{-1}(E_X) = \widetilde{E_X}$ for any $X \in S^{-1} \cdot \mathcal{J}$ and take $g, x, y \in G$.

$$W(L \otimes L)(g)W^*\xi(x, y) = (L \otimes L)(g)W^*\xi(x, xy) = W^*\xi(g^{-1}x, g^{-1}xy) =$$

$$= \xi(g^{-1}x, y) = (L \otimes 1)(g)\xi(x, y),$$

$$W(\Phi \otimes 1)(\widetilde{E_X})W^*\xi(x, y) = (\Phi \otimes 1)(\widetilde{E_X})W^*\xi(x, xy) = I_X(x)W^*\xi(x, xy) =$$

$$= I_X(x)\xi(x, y) = (\Phi \otimes 1)(\widetilde{E_X})$$

Using Lemma A.18 in [10] one can see that $(\Phi\times L)\otimes L$ is unitarily equivalent to the integrated form of the regular representation $\mathrm{Ind}\Phi$, which gives the reduced crossed product $D_\delta^{(\infty)}(S) \rtimes_{r,\tau} G$. Finally, combining this fact with unitary equivalence by operator W we deduce that $C^*(\{E_X, L_g \colon X \in S^{-1} \cdot \mathcal{J}, g \in G\})$ is isomorphic to $D_\delta^{(\infty)}(S) \rtimes_{r,\tau} G$. $\square$

Theorem 4.4. *The algebra $C_\delta^*(S)$ is isomorphic to $E_S(\pi(D_\delta^{(\infty)}(S)\rtimes_{r,\tau}G))E_S$, where π is the isomorphism defined in Lemma 4.3.*

Proof. In fact, the algebra $\mathcal{A}$ in question is the same as in Lemma 3.2, what we will now show. By Lemma 4.3, $E_S(\pi(D_\delta^{(\infty)}(S) \rtimes_{r,\tau} G))E_S$ is generated as a linear space by the operators $E_S E_{q^{-1}\cdot_G X} L_{a^{-1}} L_b E_S$ with $q, a, b \in S$, $X \in \mathcal{J}$. Such an operator can be written in another form, using the fact that $L_g E_X =$

$E_{g\cdot_G X}L_g = E_{g\cdot_G X}L_gE_S$ for all $g\in G$, $X\subset S$, and $r^{-1}\cdot_G X\cap S = r^{-1}X$ for $r\in S$, $X\in\mathcal{J}$. Using these identities we obtain:

$$E_SE_{q^{-1}\cdot_G X}L_{a^{-1}}L_bE_S = E_{q^{-1}X}L_{a^{-1}}E_{bS}L_b =$$
$$E_{q^{-1}X}E_{a^{-1}\cdot_G bS}L_{a^{-1}}L_b = E_{q^{-1}X}E_{a^{-1}bS}L_{a^{-1}b}E_S.$$

Consider the isomorphism in Lemma 3.2 and denote it by Ψ. Then by Lemma 3.2

$$E_{a^{-1}bS}L_{a^{-1}b}E_S = \Psi(T_{a^{-1}b});$$

By Lemma 3.4 $E_{q^{-1}X} = \Psi(T_{ww^{-1}})$ for some $w\in\mathcal{F}$ which depends on qX. Thus, $E_{q^{-1}X}E_{a^{-1}bS}L_{a^{-1}b}E_S\in\Psi\big(C^*_\delta(S)\big)$. From the other side, $\Psi\big(C^*_\delta(S)\big)$ is generated as a C^*-algebra by the operators $L_{a^{-1}b}E_S = \Psi(T_{a^{-1}b})$, $a,b\in S$ which are contained in $\mathcal{A}$; this shows that $\mathcal{A} = \Psi(C^*_\delta(S))$, what proves the theorem. $\square$

Theorem 4.5. *There exists a comultiplication* $\Delta\colon C^*_\delta(S)\to C^*_\delta(S)\underset{min}{\otimes} C^*_\delta(S)$, *with which* $\mathbb{Q}(S) = \big(C^*_\delta(S),\Delta\big)$ *is a compact quantum semigroup.*

Proof. The map Δ_u defined at the beginning of the Section 4 is a comultiplication on $C^*(S)$ and admits a restriction $D(S)\to D(S)\otimes_{max}D(S)$. By Lemma 3.6 $D_\delta(S)$ is isomorphic to $D(S)$. Hence $D_\delta(S)$ is endowed with a comultiplication, denote it by Δ. To see that Δ induces a limit map on $D_\delta^{(\infty)}(S)$ let us note that Δ respects the maps $\tau_{qp^{-1}}$. Indeed, for a monomial $V\in\mathcal{A}_p$ and $q\geq p$ we have

$$\Delta\tau_{qp^{-1}}(V) = \Delta(T_{qp^{-1}}VT^*_{qp^{-1}}) = T_{qp^{-1}}VT^*_{qp^{-1}}\otimes T_{qp^{-1}}VT^*_{qp^{-1}} =$$
$$\tau_{qp^{-1}}\otimes\tau_{qp^{-1}}(\Delta(V))$$

On the generators of $\pi(D_\delta^{(\infty)}(S))$, obviously Δ acts as

$$\Delta(E_{q^{-1}\cdot_G X}) = E_{q^{-1}\cdot_G X}\otimes E_{q^{-1}\cdot_G X}$$

for $q\in S, X\in\mathcal{J}$. It follows that Δ commutes with the action of G on $D_\delta^{(\infty)}(S)$ defined in (4.1). Consequently, Δ gives rise to a comultiplication on $\pi(D_\delta^{(\infty)}(S)\rtimes_{r,\tau}G)$, which we also denote by Δ. Due to the fact that $E_S\in\pi(D_\delta^{(\infty)}(S))$, using Lemma 4.4 we obtain the required comultiplication Δ on $C^*_\delta(S)$. $\square$

Remark 4.6. The bialgebras $C^*(S)$ and $C^*_\delta(S)$ are co-commutative, as for example the group C^*-algebra $C^*(G)$ of G. But their dual algebras, unlike the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$, cannot be viewed as function algebras on S or even on G. It is possible that $\phi,\psi\in(C^*_\delta(S))^*$ are nonequal but have the same values on T_a and T^*_a for all $a\in S$.

More specifically, consider $G = \mathbb{Z}$, $S = \mathbb{Z}_+$ and $\phi_k(T) = \langle T\delta_k, \delta_k\rangle$, $k \in \mathbb{Z}$. Then $\phi_k(T_a) = \phi_k(T_a^*) = \delta_0(a)$ for all $k \in \mathbb{Z}$, $a \in \mathbb{Z}_+$, but $\phi_k(T_aT_a^*) = I_{\mathbb{Z}_+}(k-a)$ while $\delta_0(T_aT_a^*) = \delta_0(a)$.

Remark 4.7. If S is abelian and has a non-empty interior in $G = S^{-1}S$ then there exists a natural short exact sequence connecting $C^*_\delta(S)$ with $C^*_\delta(G)$.

Consider the commutator ideal K in $C^*_\delta(S)$, i. e. the ideal generated by $\{[A,B] = AB-BA\colon A, B \in C^*_\delta(S)\}$. Due to Lemma 3.2, $C^*_\delta(S) = \overline{\text{lin}}\,\{E_XL_g : X \in \mathcal{J}, g \in G, X \subset gS\}$. Therefore, any generating commutator can be written as:

$$E_XL_{g_1}E_YL_{g_2} - E_YL_{g_2}E_XL_{g_1} = (E_{X\cap(g_1Y)} - E_{Y\cap(g_2X)})L_{g_1g_2}. \tag{4.2}$$

In particular, the commutator ideal contains the following operators:

$$T_aT_a^* - T_a^*T_a = E_{aS} - E_S,$$
$$T_a^*(E_X - E_S)T_a = E_{a^{-1}X} - E_S,$$
$$T_a(E_X - E_S)T_a^* = E_{aX} - E_{aS}$$

for all $a \in S$, $X \in \mathcal{J}$. Consequently, in the quotient $C^*_\delta(S)/K$ we have the equivalence classes $[E_X] = [E_S]$ for all $X \in \mathcal{J}$ and $[E_XL_g] = [E_{gS}L_g]$ for all $X\mathcal{J}$, $g \in G$ such that $X \subset gS$. Denote $\hat{L}_g = [E_{gS}L_g]$. For all $g_1, g_2 \in G$

$$\hat{L}_{g_1}\hat{L}_{g_2} = [E_{g_1S\,\cap\, g_1g_2S}L_{g_1g_2}] = \hat{L}_{g_1g_2},$$

and every $\hat{L}_g$ is unitary since $\hat{L}_g^* = \hat{L}_{g^{-1}}$.

Let us show that $\|\sum_{k=1}^n c_k\hat{L}_{g_k}\|_{C^*_\delta(S)/K} = \|\sum c_kL_{g_k}\|_{B(L^2(G))}$ for all $c_k \in \mathbb{C}$, $g_k \in G$. Since $E_{g_kS}L_{g_k} = L_{g_k}E_S$, we have

$$\|\sum_{k=1}^n c_k\hat{L}_{g_k}\|_{C^*_\delta(S)/K} \le \|\sum c_kL_{g_k}E_S\|_{B(L^2(G))} \le \|\sum c_kL_{g_k}\|_{B(L^2(G))}.$$

From the other side, the fact that S has non-empty interior implies that the norm of $T = \sum_{k=1}^n c_kL_{g_k}$ is attained on H_S. Indeed, for every $f \in L^2(G)$ and every $\epsilon > 0$ there is a compact set $K \subset G$ such that $\|f - E_Kf\| < \epsilon$; there exists [14] $g \in G$ such that $gK \subset S$, so that $E_SE_{gK} = E_{gK}$. Set $h = L_gf$. We have $E_{gK}h = L_g(E_Kf)$, and

$$\|E_Sh-h\| \le \|E_S(h-E_{gK}h)\|+\|E_SE_{gK}h-h\| \le 2\|h-E_{gK}h\| = 2\|f-E_Kf\| < 2\epsilon.$$

If f is such that $\|f\| = 1$ and $\|Tf\| > \|T\| - \epsilon$, then $\|h\| = 1$,

$$\|Th\| = \|L_gTf\| = \|Tf\| > \|T\| - \epsilon,$$

and at the same time $\|TE_Sh - Th\| \le 2\|T\|\epsilon$, what implies $\|TE_Sh\| > \|T\| - (1 + 2\|T\|)\epsilon$. This proves the statement.

Thus, we have an isomorphism $C^*_\delta(S)/K \simeq C^*_\delta(G)$ and the short exact sequence:

$$0 \to K \to C^*_\delta(S) \to C^*_\delta(G) \to 0. \tag{4.3}$$

Example 4.8. *Let us calculate the algebra $C^*_\delta(\mathbb{R}_+)$; we specify that in our notation $\mathbb{R}_+ = [0, +\infty)$.*

By our assumptions, $G = \mathbb{R}$. Below, denote $E_{[t,+\infty)} = E_t$, $t \in \mathbb{R}$. We have

$$\mathcal{J} = \{[t, +\infty) : t \in \mathbb{R}_+\} \cup \{\emptyset\},$$

and according to Lemma 3.2,

$$C^*_\delta(\mathbb{R}_+) = \overline{\mathrm{lin}}\,\{E_t L_g : t \in \mathbb{R}_+, g \in \mathbb{R}, t \geq g\}.$$

It is worth noting that

$$D(\mathbb{R}_+) = C^*(E_t : t \in \mathbb{R}_+) \subset B(L^2(\mathbb{R}_+)).$$

This algebra can be described as the space of functions supported in $\mathbb{R}_+$ and such that $\lim_{t\to t_0-0} f(t)$ exists for every $t_0 \in (0, +\infty]$ and $f(t_0) = \lim_{t\to t_0+0} f(t)$ for every $t_0 \in [0, +\infty)$. This is the uniform closure of the algebra of piecewise continuous functions, and is sometimes called by the same name.

The short exact sequence (4.3) in this case is written as

$$0 \to K \to C^*_\delta(\mathbb{R}_+) \to C^*_\delta(\mathbb{R}) \to 0.$$

By (4.2), the commutator ideal K in $C^*_\delta(\mathbb{R}_+)$ has the following form:

$$K = \overline{\mathrm{lin}}\,\{E_{[a;b)} L_g : a, b \in \mathbb{R}_+, g \in \mathbb{R}, b \geq a \geq g\}.$$

5. The semigroup von Neumann algebra

In the case when S has a non-empty interior (this does not hold automatically, see [11]), the bialgebra structure can be extended to the weak closure of $C^*_\delta(S)$ in $B(H_S)$. This closure will be denoted $VN(S)$ and termed the *semigroup von Neumann algebra*. Moreover, Δ admits a family of "multiplicative partial isometries", which play a role analogous to that of a multiplicative unitary of a locally compact quantum group.

For $d \in S$, denote $S_d = \{(a, b) \in S \times S : da^{-1}b \in S\}$, $I_d = I_{S_d}$, $I'_d = I_{(a,b):ad^{-1}b\in S}$, and $H_d = I_d \cdot L^2(S \times S)$.

Lemma 5.1. *Let $K \subset S \times S$ be a compact set. There is $d \in S$ such that $K \subset S_d$.*

Proof. Set $L = \{a^{-1}b : a, b \in K\} \subset G$. This is a compact set. Let U be an open neighbourhood of identity and $c \in S$ such that $cU \subset S$. There are $g_k \in G$, $k = 1, \dots, n$ such that $L \subset \cup_{k=1}^n g_k U$.

There is $d \in S$ such that $dg_k c^{-1} \in S$ for all k: to see this, represent g_k as $g_k = s_k t_k^{-1}$ with $s_k, t_k \in S$; since S is a reversible semigroup, $Ss_k \cap Sct_k \neq \emptyset$ for every k, so we can pick $d_k \in S$ such that $d_k s_k t_k^{-1} c^{-1} = d_k g_k c^{-1} \in S$. Next pick $d \in S$ such that $d_k \leq d$ for all k. Then we have $dd_k^{-1} \in S$, and $dg_k c^{-1} = dd_k^{-1} d_k g_k c^{-1} \in S$.

It follows that

$$dL \subset d \bigcup_k g_k U = \bigcup_k dg_k c^{-1} cU \subset \bigcup_k dg_k c^{-1} S \subset S,$$

what implies $K \subset S_d$. □

Lemma 5.2. $L^2(S \times S) = [\cup_{d \in S} H_d]$.

Proof. It follows from Lemma 5.1 that $I_K \in \cup_{d\in S} H_d$ for every compact set $K \subset S \times S$. This implies the statement immediately. □

Define now the following family of operators W_d, $d \in S$, on $L^2(S \times S)$: set $(W_d\xi)(a, b) = \xi(a, da^{-1}b)$ for every $\xi \in L^2(S \times S)$. Then $(W_d^*\xi)(a, b) = \xi(a, ad^{-1}b)$, and $W_d W_d^* = I_d$, $W_d^* W_d = I'_d$.

We claim that

$$\Delta(T)W_d = W_d(T \otimes 1)$$

for all $T \in C^*_\delta(S)$. For $T = T_c$ one verifies directly: for every $a, b \in S$, $\xi \in L^2(S \times S)$

$$\begin{aligned}\big(\Delta(T_c)W_d\xi\big)(a, b) &= \big((T_c \otimes T_c)W_d\xi\big)(a, b) = \big(W_d\xi\big)(c^{-1}a, c^{-1}b)\\ &= \xi(c^{-1}a, da^{-1}b),\\ \big(W_d(T_c \otimes 1)\xi\big)(a, b) &= \big((T_c \otimes 1)\xi\big)(a, da^{-1}b) = \xi(c^{-1}a, da^{-1}b).\end{aligned}$$

A similar calculation shows that the formula (5) is valid also for $T = T_c^*$ with $c \in S$. Since both sides of (5) are norm continuous homomorphisms, it follows that (5) holds for all $T \in C^*_\delta(S)$.

As a consequence, H_d (being the range of W_d) is invariant under $\Delta(T)$ for every $T \in C^*_\delta(S)$, $d \in S$. The identity $\Delta(T)I_d = W_d(T \otimes 1)W_d^*$ shows that $\Delta(\cdot)I_d$ is ultraweakly continuous for every d, and as such admits an extension by continuity from $C^*_\delta(S)$ to its weak closure in $B(H_S)$, which we denote by $VN(S)$ and call the *semigroup von Neumann algebra* of S. By Lemma 5.2, we have $\Delta(T) = \lim_{d\in S} \Delta(T)I_d$ (strongly) for every T. It follows (immediately by definition, since the net (I_d) is increasing) that Δ is normal.

The reasoning above results in the following

Theorem 5.3. *$VN(S)$ is a quantum semigroup with the comultiplication extended from $C^*_\delta(S)$.*

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